Analytic Solutions to Polynomial Equations

From Quadratic Formula to Hilbert's 12th Problem

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Introductory Number Theory Talk

Outline

- Number Theory Background
- My Work
  - Joseph Ferrara, **A p-adic Stark conjecture in the rank one setting**, Acta Arithmetica **193** (2020), 369-417.
  - Joseph Ferrara, **Stark's Conjectures for p-adic L-functions**, PhD thesis, UC Santa Cruz (2018).
- Applications

Algebraic Number Theory

Algebraic number theory is about *understanding* the solutions of polynomial equations with integer coefficients.

A **polynomial** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

- The a_i's are the **coefficients** of f.
- n is the **degree** of f.
- The solutions to the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$$

  are called the **roots** of f.

Polynomial Examples

Polynomial	Roots
$x^2 - 2$	$\sqrt{2}, -\sqrt{2}$
$x^2 - x - 1$	$\dfrac{1+\sqrt{2}}{2}, \dfrac{1-\sqrt{2}}{2}$
$x^2 + 1$	$\sqrt{-1}, -\sqrt{-1}$
$2x^3 - 6x + 2$	$-2\sqrt[3]{1+\sqrt{-3}}, \sqrt[3]{1+\sqrt{-3}} + \sqrt[3]{1-\sqrt{-3}}, -\sqrt[3]{1+\sqrt{-3}} + \sqrt[3]{1-\sqrt{-3}}$
$x^4 - 6x^2 + 7$	$\sqrt{3+\sqrt{2}}, -\sqrt{3+\sqrt{2}}, \sqrt{3-\sqrt{2}}, -\sqrt{3-\sqrt{2}}$

Roots of Polynomials Always Exist

Fundamental Theorem of Arithmetic

A degree n polynomial has n roots (counting repetitions) in the complex numbers. In other words, any polynomial can be factored into linear factors over the complex numbers.

A **complex number** is a number of the form $a + b\sqrt{-1}$ where a and b are real numbers.

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_{n-1})(x - \alpha_n)$$

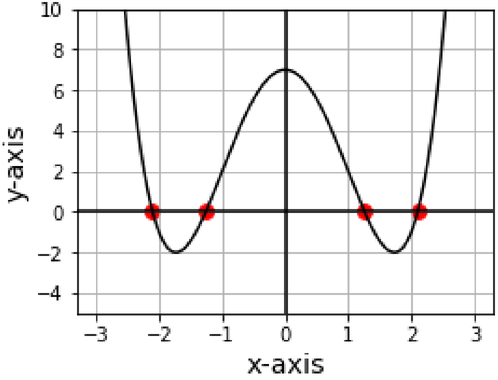
$\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Question: If every polynomial has solutions over the complex numbers, then what is the point of algebraic number theory?

Understanding Solutions to Polynomials

Answer: You want to *understand* the solutions to polynomials, not just know that they exist and be able to write them down.

$$y = x^4 - 6x^2 + 7$$



Roots Formula	Roots Decimal
$-\sqrt{3+\sqrt{2}}$	$-2.10100\ldots$
$-\sqrt{3-\sqrt{2}}$	$-1.25928\ldots$
$\sqrt{3-\sqrt{2}}$	$1.25928\ldots$
$\sqrt{3+\sqrt{2}}$	$2.10100\ldots$

General Formulas

- A formula for x in terms of the coefficients of the polynomial constitutes a *good* understanding of the solutions.

- **Quadratic Formula**: The solutions to $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- **Cubic and Quartic Formulas**: Due to Cardano and Ferrari in 1545.

- **Galois 1832**: There is no general formula for the roots of a degree 5 or higher polynomial in therms of the coefficients using only the operations of $+, -, \cdot, /$ and taking radicals $\sqrt{\ }, \sqrt[2]{\ }, \sqrt[3]{\ }, \ldots, \sqrt[n]{\ }$.
  - Further, Galois gave conditions for when a given polynomial has a formula for its roots in terms of its coefficients.
  - Example: $x^5 - x + 1 = 0$ has no formula for x.

Hilbert's 12th Problem (1900)

For certain categories of polynomials of any degree, find analytic functions that generate the roots of the polynomials.

Polynomial $\xrightarrow{(i)}$ **Infinite Sequence** $\xrightarrow{(ii)}$ **Analytic Function**

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad A_1, A_2, A_3, A_4, \ldots \qquad L(s) = \sum_{n=1}^{\infty} \frac{A_n}{n^s}$$

$\xrightarrow{(iii)}$ **Root of Polynomial**

Formula relating a value of $L(s)$ to a root of $f(x)$ (usually conjectural).

The analytic functions are known as *L-functions*.

Examples

Polynomial $\xrightarrow{(i)}$ **Infinite Sequence**

$x^2 - 4x + 1$ $\quad$ $1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0 \ldots$

$x^2 - 2x - 1$ $\quad$ $1, 0, -1, 0, -1, 0, 1, 0, 1, 0, -1, 0, -1, 0, 1, 0, 1, 0, \ldots$

$\xrightarrow{(ii)}$ **Analytic Function** $\xrightarrow{(iii)}$ **Root of Polynomial**

$$L(s) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{10^s} - \cdots \qquad L(1) = \frac{2\log(2+\sqrt{3})}{\sqrt{3}}$$

$$L(s) = 1 - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} - \frac{1}{11^s} - \frac{1}{13^s} + \cdots \qquad L(1) = \frac{\log(1+\sqrt{2})}{\sqrt{2}}$$

My Work

$$\textbf{Polynomial} \xrightarrow{(i)} \textbf{Infinite Sequence} \xrightarrow{(ii)} \textbf{Analytic Function}$$

$$\xrightarrow{(iii)} \textbf{Root of Polynomial}$$

- For two families of polynomials, I introduced a new set of analytic functions (p-adic L-functions).
- For these new analytic functions, I introduced new formulas relating them to roots of polynomials.

Applications

Theoretical

- Using p-adic L-functions could allow progress on formulas for roots of polynomials that are intractable using classical L-functions.

Applied

- New computations of solutions of polynomials.
- New computations of p-adic L-functions.
- Numeric representations of new number systems.

Polynomial Type I (Real Quadratic Fields)

$$x^{20} - 9354219385645x^{19} +$$
$$4197961532982869024\,40x^{18} - 3745796149156532581333733040x^{17} +$$
$$+500041430006230224062574451582007 0x^{16} - 213628920695837567658122371250985871454x^{15} +$$
$$-2674418326201940053665706169146045861 40x^{14} - 712620126301587970438319446258488482420x^{13} +$$
$$-732577810014457814740548349712979253855x^{12} - 981748526337760261014456925673859400785x^{11} +$$
$$-922990144478224187891067776508936658344x^{10} +$$
$$-981748526337760261014456925673859400785x^{9} - 732577810014457814740548349712979253855x^{8} +$$
$$-712620126301587970438319446258488482420x^{7} - 267441832620194005366570616914604586140x^{6} +$$
$$-213628920695837567658122371250985871454x^{5} + 5000414300062302240625744515820070x^{4} +$$
$$-3745796149156532581333733040x^{3} + 419796153298286902440x^{2} +$$
$$-9354219385645x + 1$$

Note: Coefficients are symmetric around degree 10 term.

Polynomials Type II (Imaginary Quadratic Fields)

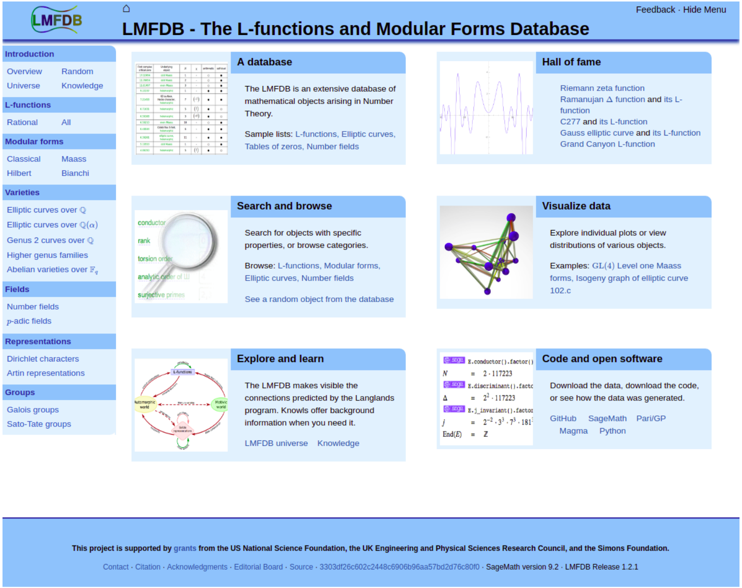
$$x^{15} - 832535x^{14} + 65231675x^{13} - 5650639400x^{12} +$$
$$+15533478425x^{11} - 39376942640x^{10} - 212804236525x^{9} - 380541320125x^{8} +$$
$$-2607229594750x^{7} - 2183192838625x^{6} + 3771011381950x^{5} - 1207366794625x^{4} +$$
$$+99067277500x^{3} - 221569375x^{2} + 466875x - 125$$

Note: All coefficients are divisible by 5.

$$x^{9} - 306x^{8} - 1143x^{7} - 71640x^{6} + 60156x^{5} + 117180x^{4} + 25704x^{3} - 7371x^{2} + 5022x - 27$$

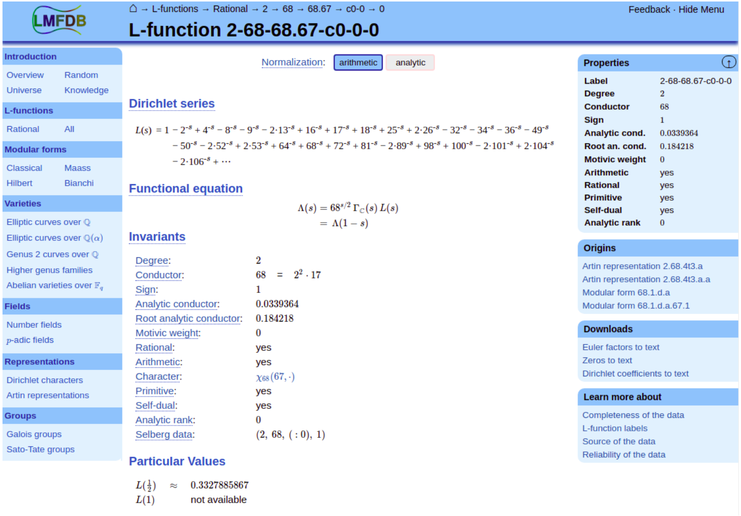
Note: All coefficients are divisible by 3.

Online Number Theory Databases



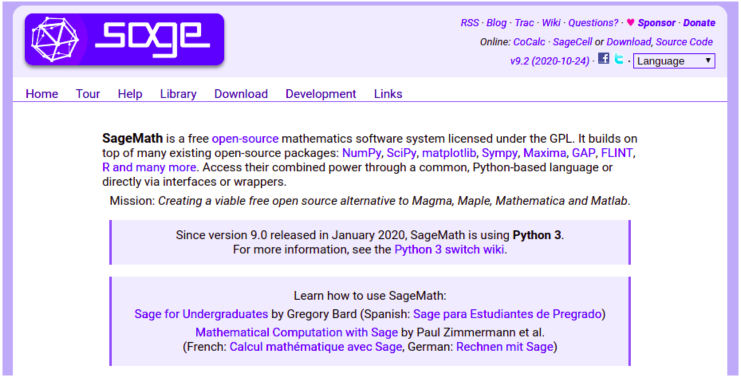
- Website: *lmfdb.org*.
  - Supported by NSF and others.
  - Maintained by mathematicians.
- I used the database in my research.
- My research will contribute to the database.

Online Number Theory Databases



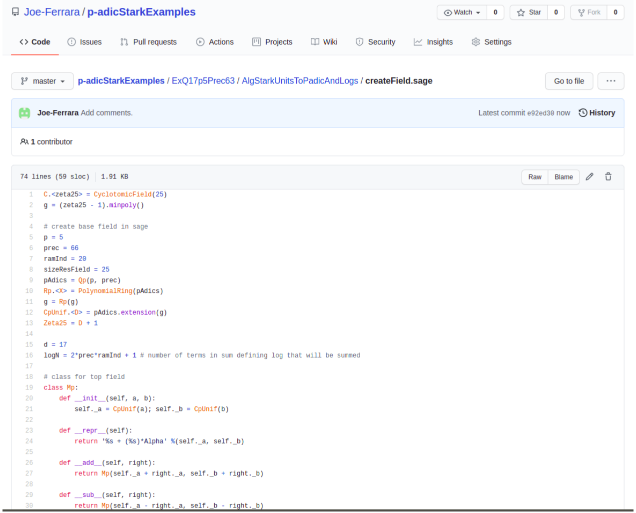
- This shows the page for the classical L-function associated to the high degree polynomial from two slides ago.
- I found solutions to this polynomial using a p-adic L-functions.
- p-adic L-functions are not in database yet.

Sage - Python Based Number Theory Libraries



- Website: *sagemath.org*.
- Open-source mathematics software.
- Sage is written and interacted with via Python.
- Contains advanced number theory libraries that contribute to research.
- Contributed to and maintained by mathematicians.

Number System Representation



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master p-adicStarkExamples / ExQ17p5Prec63 / AlgStarkUnitsToPadicAndLogs / createField.sage

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1 contributor

74 lines (59 sloc) | 1.91 KB

Raw Blame

```
C.<zeta25> = CyclotomicField(25)
g = (zeta25 - 1).minpoly()

# create base field in sage
p = 5
prec = 66
ramInd = 20
sizeResField = 25
pAdics = Qp(p, prec)
Rp.<X> = PolynomialRing(pAdics)
g = Rp(g)
CpUnif.<D> = pAdics.extension(g)
Zeta25 = D + 1

d = 17
logN = 2*prec*ramInd + 1 # number of terms in sum defining log that will be summed

# class for top field
class Mp:
    def __init__(self, a, b):
        self._a = CpUnif(a); self._b = CpUnif(b)

    def __repr__(self):
        return '%s + (%s)*Alpha' %(self._a, self._b)

    def __add__(self, right):
        return Mp(self._a + right._a, self._b + right._b)

    def __sub__(self, right):
        return Mp(self._a - right._a, self._b - right._b)
```

- I implemented a new number system in Sage.
- It is built on top of a currently implemented number system in Sage.
- The number system is used to represent p-adic L-function values and the solutions to polynomials.

References

- Joseph Ferrara, **A p-adic Stark conjecture in the rank one setting**, Acta Arithmetica **193** (2020), 369-417.
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Thank You!